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INCREMENTAL ANALYSIS OF LARGE DEFLECTIONS OF SHELLS OF REVOLUTION

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Abstract—By beginning with a general incremental variational theory for the analysis of geometrically and physically nonlinear problems in continuum mechanics, the development is specialized for an elastic continuum. Then the deduced variational expression is re-cast into the displacement form of the finite element method of analysis and specialized for the large displacement analysis of elastic shells of revolution. The convergence and accuracy of the developed procedure are studied on a very sensitive numerical example. The general incremental theory presented has a wide range of applicability and goes beyond the special solution of this paper.

INTRODUCTION

THE incremental method of analysis in continuum mechanics is closely related to the developments in the theory of the continuum which is under initial stress. Following the original work of Cauchy [1], the superposition of small displacements or displacement gradients on large deformations of an elastic body under initial stress has received much attention by many investigators; however, not all of them have arrived at the same results. The differences arise mostly by the introductions of simplifying assumptions and the method of linearization of the superposed deformations. Cauchy's constitutive equations were rederived by Murnaghan [2] who assumes Green's elasticity and superposes infinitesimal displacement gradients on large initial deformations. A more restrictive formulation in which superposed displacements themselves are infinitesimal was developed by Green *et al.* [3]. All of these results indicate that in the incremental constitutive equations the isotropy of an initially isotropic material is lost even if the displacement gradients of the superposed displacements are infinitesimal. In both [2] and [3] the linearization of the superposed deformations is performed by simple mathematical perturbation of the constitutive equations and the detailed study of the superposed deformation is bypassed.

In order to study more closely the nature of the superposed deformations, a group of investigators have used the idea of polar decomposition of the strain tensor by separating it into pure deformations and rotations. Then constitutive equations involving only pure deformation can be written and the separation of the physics and geometry of the incremental deformations can be achieved. This approach has been followed by Biezeno and Hencky [4] and has been extensively discussed and used by Biot [5]. A parallel and more restricted development was reported by Prager [6] who assumes that the incremental constitutive equations are linear and isotropic. In general the strain measure of pure deformation cannot be expressed as a rational function of the displacement gradients unless the linear strains are assumed infinitesimal. After such approximation this measure of strain is no longer a tensor and it is difficult to use in curvilinear coordinates.

The increasing interest in nonlinear analysis of structures has accelerated the application of the incremental method of analysis to such problems. In an attempt to extend Biot's ideas to the analysis of large deformations, Felippa [7] wrote an expression of virtual work in which he uses the Lagrange strain tensor together with Biezeno–Hencky type of stress. This stress and strain are not conjugate in the sense that their product does not represent work unless the deformations are infinitesimal and rotations are first order. The use of the incremental procedure together with the finite element technique is gaining momentum in the analysis of nonlinear problems in structural analysis [8–12]. Therefore, a rigorous approach for the solution of such problems is much needed and this paper attempts to clarify some of the basic concepts.

First, a general incremental variational formulation for the analysis of the geometrically and physically nonlinear problems in continuum mechanics is presented. The incremental variational expression of the equilibrium of the continuum is based on a moving reference configuration. Then, starting from the laws of thermodynamics, the incremental nonlinear constitutive equations of elasticity are derived in such a manner that they can be used in the variational expressions. The problem is then re-cast into the form of the displacement formulation of the finite element method which is then specialized for the analysis of elastic shells of revolution. The convergence and the high degree of accuracy of the developed procedure are illustrated on a numerical example of a thin-walled torus subjected to external pressure.

THE PRINCIPLE OF VIRTUAL WORK

Consider two configurations of a deformable body on its path of deformation from an initial state characterized by at most an isotropic state of stress to a final configuration (see Fig. 1). These are called configurations 1 and 2. The volume, boundary surface and the coordinates of the material points of the body in the initial, first and second configurations are denoted by $\bar{v}$, $\bar{a}$, $\bar{z}_i$, v, a, z_i and V, A, Z_i respectively.

A generic point $\bar{p}$ in the initial state will occupy positions p and P in configurations 1 and 2, respectively. The displacement vectors between these positions are shown in Fig. 1. The equations of equilibrium at configurations 1 and 2 can be written in the form of the expressions of virtual work in different ways depending on the choice of the reference configuration for the variables involved, and also on the vectors in terms of which the virtual displacements are expressed. For example, the variables in configuration 1 can be written with reference to the coordinates of any configuration desired; also the virtual displacements for point p can be written as $\delta({}^1\mathbf{u})$ or $\delta\mathbf{u}$ and for point P can be written $\delta\mathbf{u}$ or $\delta({}^2\mathbf{u})$. A variational expression in which configuration 1 is taken as the reference and $\delta\mathbf{u}$ as the virtual displacement is derived in this presentation and used in the subsequent developments. Two alternative formulations of virtual work have been derived in [13]. The expression of virtual work W_v at configuration 2 is

$$W_v = \int_A \mathbf{T} \,.\, \delta\mathbf{u} \,\mathrm{d}A + \int_V \rho\mathbf{F} \,.\, \delta\mathbf{u} \,\mathrm{d}V \tag{1}$$

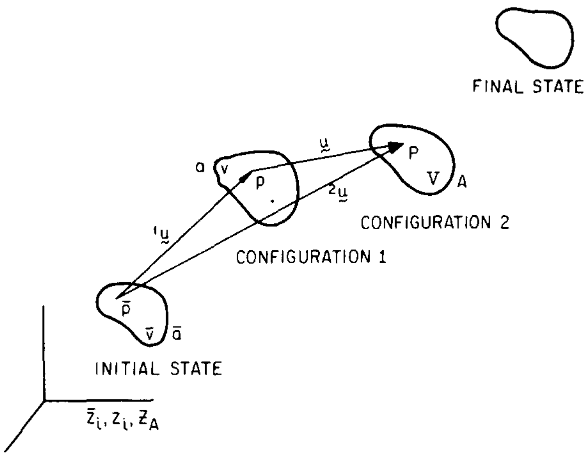

FIG. 1.

where **T** is the surface traction per unit of area A, ρ is the mass density in configuration 2 and **F** is the body force per unit of mass.

Equation (1) can be written in terms of the coordinates of configuration 1 by choosing proper definitions for traction and body force. One such traction is defined by

$$^{2}\mathbf{t} = \mathbf{T}\frac{\mathrm{d}A}{\mathrm{d}a} \tag{2}$$

where $^{2}\mathbf{t}$ is the traction in configuration 2 and measured per unit of area in configuration 1 and da is the element of surface area in configuration 1.

The stresses associated with traction $^{2}\mathbf{t}$ can be defined in various ways, one of which is the symmetric Piola stress tensor [14]. Consider the neighborhood of the point p in configuration 1 and the same neighborhood in configuration 2. For simplicity of presentation a two dimensional picture of such neighborhood is shown in Fig. 2, although the

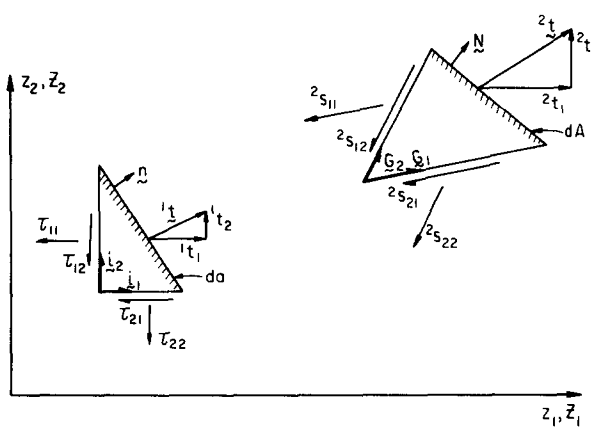

FIG. 2.

theoretical development is carried out for a three dimensional body. The Cauchy stresses in this neighborhood in configuration 1 are τ_{ij} which are associated with the unit base vectors $\mathbf{i}_k$. The Piola symmetric stresses of our interest which act in the same material neighborhood in configuration 2 are called ${}^2s_{ij}$. These stresses are associated with the deformed base vectors $\mathbf{G}_i$. For example ${}^2s_{mn}$ denotes a force acting in configuration 2 on a surface which had a unit area in configuration 1 and was perpendicular to base vector $\mathbf{i}_m$, and which is parallel to the base vector $\mathbf{G}_n$. Therefore, these stresses are in configuration 2 but are measured per unit area in configuration 1. The relationship between ${}^2\mathbf{t}$ and the stresses ${}^2s_{ij}$ is [14]

$$ {}^2t_k = {}^2s_{ij}\frac{\partial Z_k}{\partial z_j}n_i \tag{3} $$

where $\mathbf{n}$ is the unit normal vector to surface $\mathbf{a}$.

Considering the law of conservation of mass, the second integral on the right hand side of equation (1) can be written as

$$ \int_V \rho\mathbf{F}\,.\,\delta\mathbf{u}\,\mathrm{d}V = \int_v \rho_0\,{}^2f_k\delta u_k\,\mathrm{d}v \tag{4} $$

where ${}^2f_k$ denotes the body force per unit mass acting in configuration 2, but measured in terms of the coordinates in configuration 1. Substitution of (2)–(4) into (1) and transformation of the surface integral to equivalent volume integrals yields

$$ W_v = \int_v {}^2s_{ij}\delta\varepsilon_{ij}\,\mathrm{d}v + \int_v [({}^2s_{ij}Z_{k,j})_{,i} + \rho_0\,{}^2f_k]\delta u_k\,\mathrm{d}v \tag{5} $$

in which ε_{ij} is the Lagrangian strain from configuration 1 to 2, and is given by

$$ \varepsilon_{ij} = \tfrac{1}{2}(u_{i,j} + u_{j,i} + u_{k,i}u_{k,j}). \tag{6} $$

The integrand in the second integral of equation (5) is the expression for the equilibrium of the body and is equal to zero [14].

The magnitude of the components of stress tensor ${}^2s_{ij}$ can be arbitrarily divided into two parts (see Fig. 3).

$$ {}^2s_{ij} = \tau_{ij} + s_{ij} \tag{7} $$

in which τ_{ij} have the same magnitude as the corresponding Cauchy stresses in configuration 1 but are associated with the base vectors $\mathbf{G}$, and s_{ij} are symmetric stress components

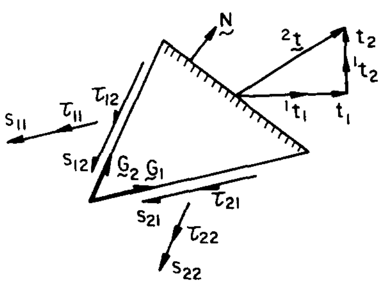

FIG. 3.

which have magnitudes equal to the difference between the stresses ${}^2s_{ij}$ and τ_{ij}. Substitution of equation (7) into (5) results in

$$\int_a {}^2t_i\delta u_i \,\mathrm{d}a + \int_v \rho_0 \,{}^2f_i\delta u_i \,\mathrm{d}v = \int_v (\tau_{ij}+s_{ij})\delta\varepsilon_{ij} \,\mathrm{d}v \tag{8}$$

The expression for virtual work at configuration 1 can be written as

$$\int_a {}^1t_i\delta u_i \,\mathrm{d}a + \int_v \rho_0 \,{}^1f_i\delta u_i \,\mathrm{d}v = \int_v \tau_{ij}\delta e_{ij} \,\mathrm{d}v \tag{9}$$

in which ${}^1t_i$ is the traction acting per unit of area of a, ${}^1f_i$ is the body force acting per unit mass in configuration 1, and e_{ij} is the linear part of Lagrangian strain between configurations 1 and 2.

$$e_{ij} = \tfrac{1}{2}(u_{i,j}+u_{j,i}). \tag{10}$$

Subtraction of equation (9) from (8) gives

$$\int_a t_i\delta u_i \,\mathrm{d}a + \int_v \rho_0 f_i\delta u_i \,\mathrm{d}v = \int_v (\tau_{ij}\delta\eta_{ij}+s_{ij}\delta\varepsilon_{ij}) \,\mathrm{d}v \tag{11}$$

where

$$\eta_{ij} = \varepsilon_{ij}-e_{ij} = \tfrac{1}{2}u_{k,i}u_{k,j} \tag{12}$$

is the nonlinear part of the increment of Lagrangian strain between configurations 1 and 2,

$$t_i = {}^2t_i - {}^1t_i,$$

and

$$f_i = {}^2f_i - {}^1f_i. \tag{13}$$

This is the incremental expression of virtual work which in effect is a statement of the equilibrium equations of the body at configuration 2 in terms of the variables which are expressed in the coordinates of configuration 1. The proof that equation (11) leads to the incremental equilibrium equations and the corresponding boundary conditions is given in Ref. [13] where the principle of virtual work is derived in curvilinear coordinates.

INCREMENTAL CONSTITUTIVE EQUATIONS OF ELASTICITY

The constitutive equations of the continuum are written based on mathematical approximations of physical observations subject to the laws of thermodynamics, and some invariance requirements like the principle of material frame indifference [15, 16]. For an elastic continuum the constitutive equations in a variable configuration 1 are

$${}^1s_{ij} = \bar{\rho}_0 \frac{\partial\,{}^1A}{\partial\,{}^1\varepsilon_{ij}}, \tag{14}$$

$${}^1S = -\frac{\partial\,{}^1A}{\partial\theta}, \tag{15}$$

and

$$\bar{q}_k \theta_{,k} \geq 0 \tag{16}$$

where ${}^1s_{ij}$ is the Piola symmetric stress tensor in configuration 1 measured per unit area $\bar{a}$, ${}^1\varepsilon_{ij}$ is the Lagrangian strain tensor between the initial state and configuration 1, ${}^1A$ is the free energy function, ${}^1S$ is the entropy per unit mass, θ is temperature, $\bar{q}_k$ is the rate of heat flux in configuration 1 across a convected coordinate surface which in the initial state is perpendicular to base vector $\mathbf{i}_k$ and $\bar{\rho}_0$ is the mass density in the initial state.

The laws of thermodynamics for a variable configuration 2 can be written as

$$\bar{\rho}_0({}^2r) - \bar{\rho}_0({}^2\dot{A} + {}^2\dot{S}\theta + {}^2S\dot{\theta}) - {}^2\bar{q}_{k,k} - {}^2\bar{s}_{kl}\,{}^2\dot{\varepsilon}_{kl} = 0, \tag{17}$$

and

$$-\bar{\rho}_0({}^2\dot{A} + {}^2S\dot{\theta}) + {}^2\bar{s}_{kl}\,{}^2\dot{\varepsilon}_{kl} - \frac{{}^2\bar{q}_k \theta_{,k}}{\theta} \geq 0 \tag{18}$$

in which ${}^2r$ is the rate of heat production, and ${}^2\bar{s}_{ij}$ is the Piola symmetric stress tensor in configuration 2 measured per unit of area $\bar{a}$. It is possible to divide the functions in equations (17) and (18) into two parts

$$\begin{aligned}
{}^2A &= {}^1A({}^1\varepsilon_{ij}, \theta) + A({}^1\varepsilon_{ij}, \xi_{ij}, \theta) \\
{}^2S &= {}^1S({}^1\varepsilon_{ij}, \theta) + S({}^1\varepsilon_{ij}, \xi_{ij}, \theta) \\
{}^2\bar{s}_{ij} &= {}^1s_{ij}({}^1\varepsilon_{ij}, \theta) + \bar{s}_{ij}({}^1\varepsilon_{ij}, \xi_{ij}, \theta) \\
{}^2\bar{q}_i &= {}^1\bar{q}_i({}^1\varepsilon_{kl}, \theta, \theta_{,k}) + \bar{q}_i({}^1\varepsilon_{kl}, \xi_{kl}, \theta, \theta_{,k}) \\
{}^2r &= {}^1r({}^1\varepsilon_{kl}, \theta, \theta_{,k}) + r({}^1\varepsilon_{kl}, \xi_{kl}, \theta, \theta_{,k})
\end{aligned} \tag{19}$$

where

$${}^2\varepsilon_{ij} = {}^1\varepsilon_{ij} + \xi_{ij} \tag{20}$$

and ξ_{ij} can be shown to be

$$\xi_{ij} = \frac{\partial z_k}{\partial \bar{z}_i}\frac{\partial z_e}{\partial \bar{z}_j}\varepsilon_{kl}. \tag{21}$$

Substitution of equations (19), (15) and (14) into (18) results in

$$\left({}^2\bar{s}_{kl} - \bar{\rho}_0\frac{\partial A}{\partial \xi_{kl}}\right)\dot{\xi}_{kl} + \left(\bar{s}_{kl} - \bar{\rho}_0\frac{\partial A}{\partial^1\varepsilon_{kl}}\right){}^1\dot{\varepsilon}_{kl} - \bar{\rho}_0\left(S + \frac{\partial A}{\partial \theta}\right)\dot{\theta} - \frac{{}^2\bar{q}_k\theta_{,k}}{\theta} \geq 0. \tag{22}$$

In the same manner the energy equality equation (17) reduces to

$$\bar{\rho}_0 r + \rho_0(A + \dot{\theta}S + \theta\dot{S}) - \bar{q}_{k,k} - {}^2\bar{s}_{kl}\dot{\xi}_{kl} - \bar{s}_{kl}\,{}^1\dot{\varepsilon}_{kl} = 0. \tag{23}$$

Since ${}^1\dot{\varepsilon}_{kl}$, $\dot{\xi}_{kl}$ and $\dot{\theta}$ can be chosen arbitrarily, following a similar argument presented by Coleman and Noll [16] it can be concluded that

$$\bar{s}_{kl} = \bar{\rho}_0\frac{\partial A}{\partial^1\varepsilon_{kl}}, \tag{24}$$

$${}^2\bar{s}_{kl} = \bar{\rho}_0\frac{\partial A}{\partial \xi_{kl}}, \tag{25}$$

$$S = -\frac{\partial A}{\partial \theta}, \tag{26}$$

and

$$^2\bar{q}_k\theta_{,k} \geq 0. \tag{27}$$

These are the incremental constitutive equations for an elastic continuum.

All the discussion in this section has dealt with stresses $^1\bar{s}_{ij}$, $^2\bar{s}_{ij}$ and $\bar{s}_{ij}$ which are measured per unit of area $\bar{a}$ in the initial state. In order to be able to use the constitutive equations (24) and (25) in the expression of virtual work equation (11), they must be expressed in terms of stresses $^2s_{ij}$ and s_{ij} which are measured per unit of area a in configuration 1. The following transformations hold between the Cauchy stress tensor in configuration 2 and the Piola stresses $^2s_{ij}$ and $^2s_{ij}$ [14].

$$^2s_{ij} = \frac{\rho_0}{\rho}\frac{\partial z_i}{\partial Z_M}\frac{\partial z_j}{\partial Z_N}\tau_{MN}, \tag{28}$$

$$\tau_{MN} = \frac{\rho}{\bar{\rho}_0}\frac{\partial Z_M}{\partial \bar{z}_m}\frac{\partial Z_N}{\partial \bar{z}_n}{}^2\bar{s}_{mn}. \tag{29}$$

Substitution of equation (29) into (28) gives

$$^2s_{ij} = \frac{\rho_0}{\bar{\rho}_0} z_{i,m} z_{j,n}\, {}^2\bar{s}_{mn}, \tag{30}$$

and also substitution of equations (7) and (19) into (30) results in

$$s_{ij} = \frac{\rho_0}{\bar{\rho}_0} z_{i,m} z_{j,n} \bar{s}_{mn}. \tag{31}$$

The incremental constitutive equations in terms of $^2s_{ij}$ and s_{ij} can be obtained by substituting equations (30) and (31) into (24) and (25).

$$^2s_{ij} = \rho_0 z_{i,m} z_{j,n} \frac{\partial A}{\partial \xi_{mn}} \tag{32}$$

$$s_{ij} = \rho_0 z_{i,m} z_{j,n} \frac{\partial A}{\partial\, {}^1\varepsilon_{mn}}. \tag{33}$$

The constitutive equations (24), (25), (32) and (33) are quite general. If the free energy function A is known in terms of $^1\varepsilon_{kl}$ and ξ_{kl} then $\bar{s}_{kl}$ and $^2\bar{s}_{kl}$ can be determined. Assuming that A is an analytic function it can be expanded as a power series of $^1\varepsilon_{kl}$ and ξ_{kl}. In particular for isotropic materials A can be expressed as a function of the invariants of $^1\varepsilon_{kl}$ and ξ_{kl}. For certain problems a finite number of terms in the power series expansion is enough to approximate A and hence the constitutive equations accurately. In the special case where deformations are infinitesimal but displacements and rotations are not, the retention of the terms in the power series of A up to the second power of $^1\varepsilon_{kl}$ and ξ_{kl} is sufficient. Then it is shown in [13] that equations (24) and (25) become

$$\bar{s}_{kl} = \lambda\delta_{kl}\xi_{ii} + 2\mu\xi_{kl}, \tag{34}$$

and

$$^2\bar{s}_{kl} = \lambda\delta_{kl}({}^2\varepsilon_{ii}) + 2\mu({}^2\varepsilon_{kl}) \tag{35}$$

where λ and μ are the elasticity constants. The constitutive equations (32) and (33) become

$$^{2}s_{ij} = \frac{\rho_0}{\bar{\rho}_0} z_{i,k} z_{j,l} [\lambda \delta_{kl} (^{2}\varepsilon_{ii}) + 2\mu (^{2}\varepsilon_{kl})] \tag{36}$$

and

$$s_{ij} = \frac{\rho_0}{\bar{\rho}_0} z_{i,k} z_{j,l} [\lambda \delta_{kl} \xi_{ii} + 2\mu \xi_{kl}]. \tag{37}$$

DISPLACEMENT FORMULATION FOR A NONLINEAR INCREMENTAL PROCEDURE

The incremental expression of virtual work in configuration 2 is given by equation (11). Assume that the material space of the body in configuration 1 is composed of a set of simply connected subregions called finite elements. Then equation (11) can be thought of as the sum of similar expressions for the elements. Let the displacement increments, $u_i(z_m)$, of the points in the elements be expressed in terms of the displacement increments, $r_j(z_n)$, of certain points or sets of points of the elements called the nodes by some interpolation functions $M_{ij}(z_m)$ as

$$u_i(z_m) = M_{ij}(z_m) r_j(z)_n. \tag{38}$$

The displacements $u_i(z_m)$ are continuous in the element and vanish beyond the boundaries of the element. Thus the element is the support for the functions $u_i(z_m)$. The combination of all such displacements for all the elements comprise the total incremental displacement field of the whole body. The element displacement and geometry representation should be such that the rigid body motion of the elements and the compatibility requirements at the element boundaries be satisfied. In addition, for the uniform convergence of solutions, the displacements must be such that uniform straining modes of the elements exist [17–19].

In the same way the incremental tractions and body forces are expressed in terms of the magnitude of tractions and body forces at the nodes by some interpolation functions such as

$$t_i(z_k) = {}^{t}M_{im}(z_k) T_m(z_n) \tag{39}$$

$$f_i(z_m) = 4M_{ij}(z_m) F_j(z_n). \tag{40}$$

For materials where the relationship between s_{ij} and ε_{ij} in the elements can be expressed by†

$$s_{ij} = C_{ijkl} \varepsilon_{kl}, \tag{41}$$

the substitution of equations (38)–(41) into (11) results in the following incremental force-displacement relationship:

$$R_k = (K_{kn}^{(0)} + K_{kn}^{(1)}) r_n + \underline{K_{kns}^{(2)} r_n r_s + K_{knsu}^{(3)} r_n r_s r_u} \tag{42}$$

† When (33) is used more complicated, implicit nonlinear incremental equations are obtained. The stated relationship is sufficiently general for our purpose here.

where

$$R_k = \int_a {}^t M_{im} M_{ik} T_m \, da + \int_v \rho_0 \, {}^f M_{im} M_{ik} F_m \, dv, \tag{43}$$

$$K^{(0)}_{kn} = \frac{1}{4} \int_v C_{ijrt} (M_{ik,j} + M_{jk,i})(M_{rn,t} + M_{tn,r}) \, dv, \tag{44}$$

$$K^{(1)}_{kn} = \frac{1}{2} \int_v \tau_{ij} (M_{mk,i} M_{mn,j} + M_{mn,i} M_{mk,j}) \, dv, \tag{45}$$

$$K^{(2)}_{kns} = \frac{1}{4} \int_v C_{ijrt} [(M_{ik,j} + M_{jk,i}) M_{ms,r} M_{mn,t} + (M_{rs,t} + M_{ts,r})(M_{mk,i} M_{mn,j} + M_{mk,j} M_{mn,i})] \, dv, \tag{46}$$

and

$$K^{(3)}_{knsu} = \frac{1}{4} \int_v C_{ijrt} M_{ms,r} M_{mu,t} (M_{mk,i} M_{mn,j} + M_{mk,j} M_{mn,i}) \, dv. \tag{47}$$

The stiffness matrix $K^{(0)}_{kn}$ is due to the linear part of $s_{ij}\delta\varepsilon_{ij}$ in equation (11). The stiffness matrix $K^{(1)}_{kn}$, called the tangent or initial stress stiffness matrix, is due to the $\tau_{ij}\delta\eta_{ij}$ term in equation (11), and the matrices $K^{(2)}_{kns}$ and $K^{(3)}_{knsu}$ are due to the nonlinear terms in $s_{ij}\delta\varepsilon_{ij}$. It can be seen that the incremental force–displacement relationship in equation (42) consists of a linear part and an underlined nonlinear part. In general, this relationship must be solved by successive approximations, e.g. iteration. The linear part of equation (42) can be used as a first approximation in a direct incremental procedure.

When the nodal displacement increments are found from equation (42) they are added to the total nodal displacements at configuration 1 to give the total nodal displacements at configuration 2

$${}^2 r_j(z_n) = {}^1 r_j(z_n) + r_j(z_n). \tag{48}$$

Substitution of $r_j(z_n)$ in equation (38) and the result in equation (41) gives the increments of stresses s_{ij}. Then by equations (7), (30) and (29) the Cauchy stress components in configuration 2 are found.

ANALYSIS OF SHELLS OF REVOLUTION

The accuracy and convergence of the incremental method developed in the preceding sections will be demonstrated by the finite element analysis of large displacements of axisymmetric shells of revolution. Further study in this area is reported in [13] where the comparison of the results for circular plates with the solution in [20] and for the shallow shells with the solutions in [21, 22] showed good agreement.

For the finite element analysis an axisymmetric curved shell element is used. The element employs a substitute curve for the shell surface which passes through two external nodal points i and j, and which has the same slope at these points as the shell meridional curve [see Fig. 4(a)]. The main features of this element are the same as the one devised in [23] except that it has two additional internal nodal points [see Fig. 4(b)]. The equation

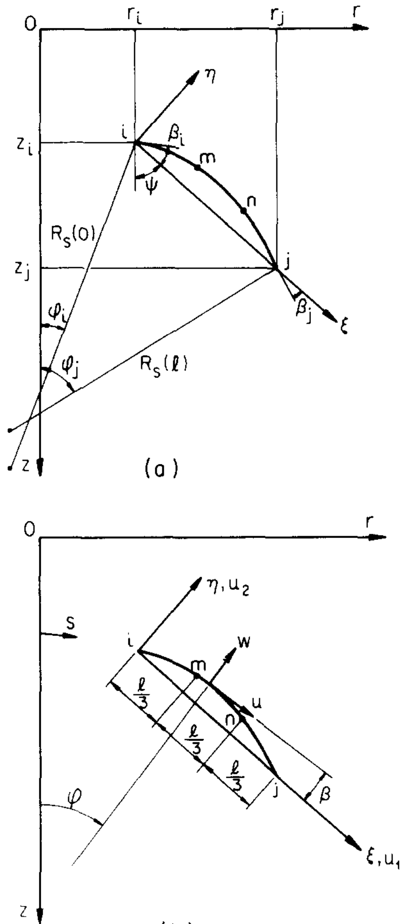

FIG. 4.

of the element curve is given by

$$\eta = \xi(1 - \xi)(a_1 + a_2\xi) \tag{49}$$

where

$$a_1 = \tan \beta_i, \qquad a_2 = \tan \beta_i + \frac{1}{2}\left(\frac{d^2\eta}{d\xi^2}\right)_{\xi=0}. \tag{50}$$

The displacement increment vector between configurations 1 and 2 for the middle surface of the shell can be decomposed into components u_1 and u_2 which are parallel to ξ and η axes or w and u which are normal and tangential to the meridional curve in configuration 1 (see Fig. 5). In both [23] and [13] u_1 and u_2 were expressed by first and third order polynomials of ξ respectively. In the present work a third order polynomial expansion is used for u_1 as well as u_2. This is found to increase the accuracy of results appreciably whenever there are sharp gradients of the meridional bending moments in the shells which

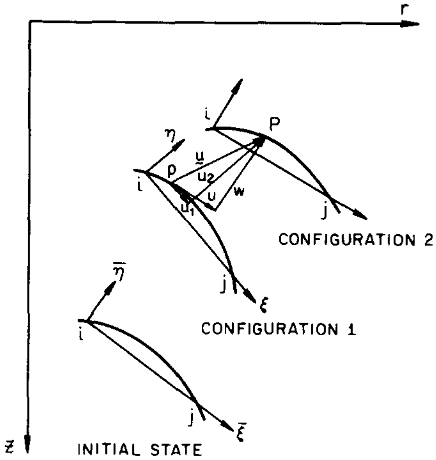

FIG. 5.

is made of a material which possesses a Poisson ratio other than zero [24]. Thus

$$\begin{aligned} u_1 &= \alpha_1+\alpha_2\xi+\alpha_3\xi^2+\alpha_4\xi^3 \\ u_2 &= \alpha_5+\alpha_6\xi+\alpha_7\xi^2+\alpha_8\xi^3. \end{aligned} \tag{51}$$

Since there are only six degrees of freedom at the external nodal points i and j for axisymmetrically deforming shells of revolution, then corresponding to the two extra independent constants in equation $(51)_1$, two internal nodes m and n each with one degree of freedom along ξ are introduced at $\xi = l/3$ and $\xi = 2l/3$ as shown in Fig. 4(b). These nodal points are eliminated in the process of condensation of the element stiffness matrix.

STRAIN–DISPLACEMENT RELATIONS

The physical components of the Lagrange strain tensor between configuration 1 and 2 and in the neighborhood of a generic point p of the shell whose deformation is subject to Kirchoff are given by

$$\begin{aligned} \varepsilon_{ss} &= \varepsilon_s+\zeta\varkappa_s \\ \varepsilon_{\theta\theta} &= \varepsilon_\theta+\zeta\varkappa_\theta \end{aligned} \tag{52}$$

where ε_s, ε_θ and $\varkappa_s$, $\varkappa_\theta$ are the middle surface strains and changes of curvatures respectively. It is shown in [13] that for infinitesimal deformations but finite rotations and displacements, the quantities in equation (52) are given by

$$\begin{aligned} \varepsilon_s &= e_s+\frac{\chi^2}{2}, \qquad & \varkappa_s &= \chi_{,s}+\frac{e_s}{R_s}+\frac{\chi^2}{2R_s} \\ \varepsilon_\theta &= e_\theta, \qquad & \varkappa_\theta &= \frac{\cos\varphi}{r}\chi+\frac{e_\theta}{R_\theta}-\frac{\sin\varphi}{2r}\chi^2 \end{aligned} \tag{53}$$

where

$$e_s = u_{,s} + \frac{w}{R_s}, \qquad e_\theta = \frac{1}{r}(u\cos\varphi + w\sin\varphi)$$
$$\chi = \frac{u}{R_s} - w_{,s}. \tag{54}$$

CONSTITUTIVE EQUATIONS

For initially isotropic elastic material the incremental constitutive relations equations (37) reduce to the following form for axisymmetrically deforming shells of revolution [13].

$$\begin{Bmatrix} s_{11} \\ s_{22} \end{Bmatrix} = \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix} \begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \end{Bmatrix} \tag{55}$$

where

$$C_{11} = \frac{A^3}{B}\frac{E}{1-\nu^2}$$
$$C_{12} = C_{21} = AB\frac{\nu E}{1-\nu^2} \tag{56}$$
$$C_{22} = \frac{B^3}{A}\frac{E}{1-\nu^2}$$
$$A = \sqrt{[1+2({}^1\varepsilon_{11})]}, \qquad B = \sqrt{[1+2({}^1\varepsilon_{22})]}.$$

When the linear part of the incremental equation (42) is used then only the linear parts of the incremental strains must be substituted into equation (55).

THE STIFFNESS MATRIX

For an element of the axisymmetric shell of revolution the linear part of the incremental force–displacement relation equation (42) becomes

$$\underset{8\times 1}{\{R\}} = \underset{8\times 1}{[k]} \quad \underset{8\times 1}{\{r\}} \tag{57}$$

where the stiffness matrix $[k]$ consists of two parts $k^{(0)}$ and $k^{(1)}$ each given by

$$k^{(0)} = \int_0^1 l([B]^T[D][B])r(1+\eta'^2)^{\frac{1}{2}}\,d\xi,$$
$$k^{(1)} = \int_0^1 l([G]^T[F][N][G])r(1+\eta'^2)^{\frac{1}{2}}\,d\xi. \tag{58}$$

The matrices in equation (58) are given in Appendix A. By the known procedure of condensation [27] matrix $[k]$ is reduced to a 6×6 matrix which matches the available degrees of freedom at the external nodes. Then equation (57) becomes

$$\underset{6\times 1}{\{\bar{R}\}} = \underset{6\times 6}{[\bar{k}]} \underset{6\times 1}{\{\bar{r}\}}. \tag{59}$$

The reduced stiffness matrices of the elements are then assembled by the direct stiffness matrix method [25] to find the incremental force–displacement relation of the whole structure.

NUMERICAL EXAMPLE

The behavior of the complete torus in Fig. 6 is studied. The torus is under external pressure and its geometrical, loading and elastic properties are $L/R = 1{\cdot}5$, $h/R = 0{\cdot}01$, $p/E = 10^{-5}$ and Poisson's ratio $\nu = 0{\cdot}3$. The results of both linear and nonlinear analyses

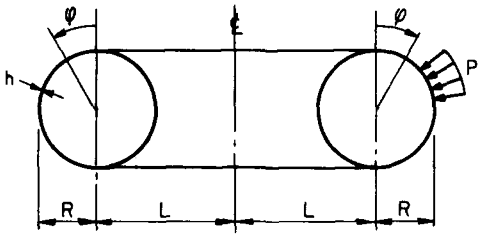

FIG. 6.

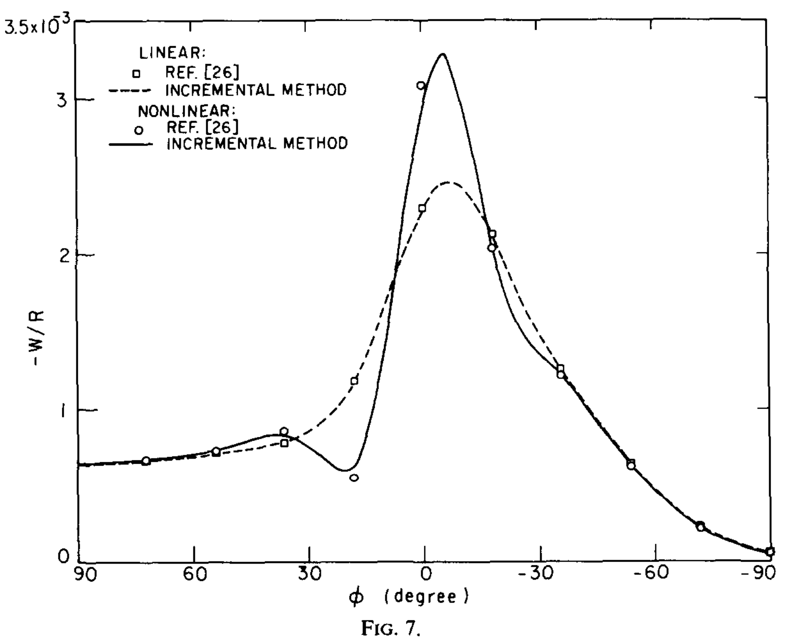

FIG. 7.

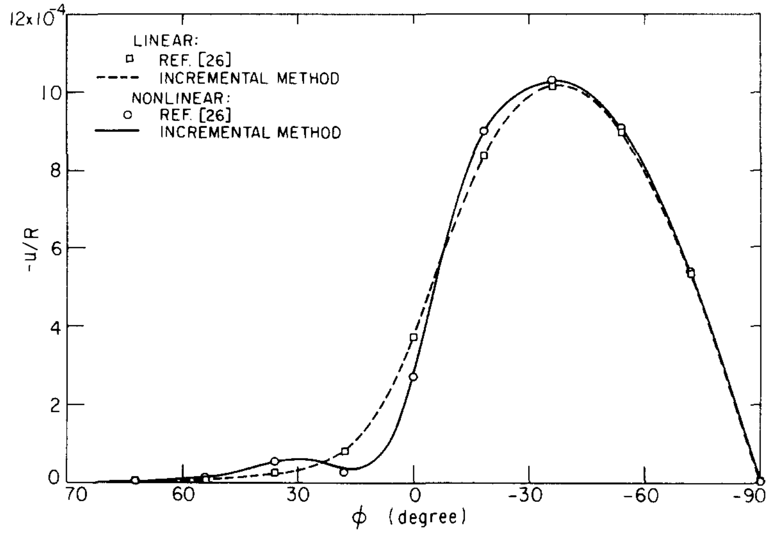

FIG. 8.

are plotted in Figs. 7–11. The significant difference between the stress resultants and normal displacements in the linear and nonlinear solution is an indication of the highly nonlinear characteristic of the torus and the necessity of a nonlinear analysis even at low values of the external load. The same structure has been analyzed by Kalnins and Lestingi

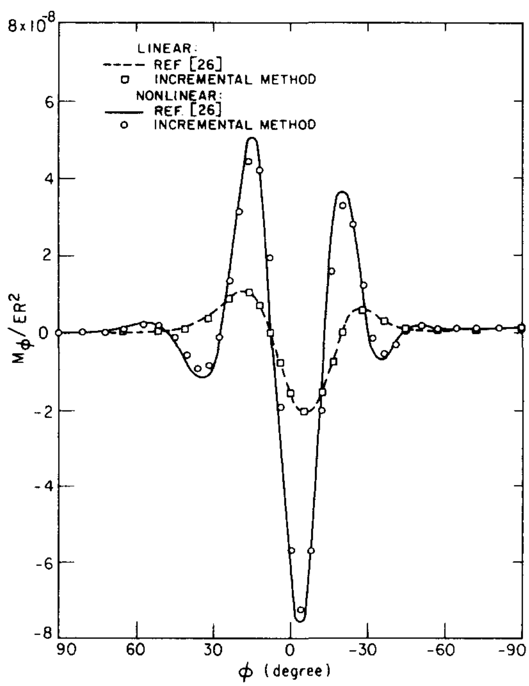

FIG. 9.

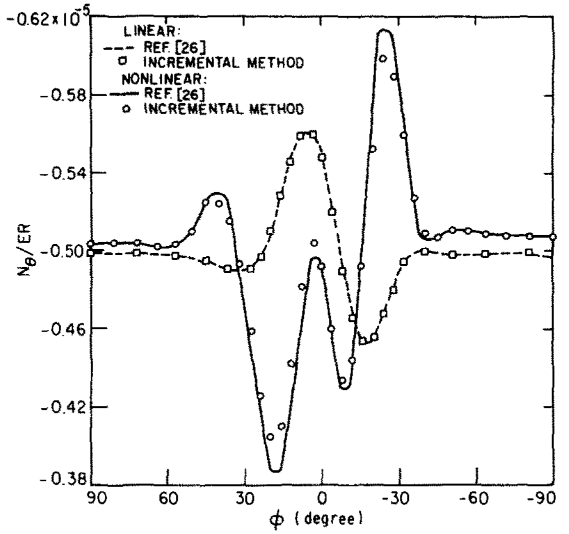

FIG. 10.

[26] who have changed the boundary value problem of the shell into an initial value problem and have used Adams procedure. The good agreement of the present results with their solution, as exhibited in Figs. 7–11, shows that even with the linear part of equation (42) and for small load increments reasonably accurate solutions can be achieved.

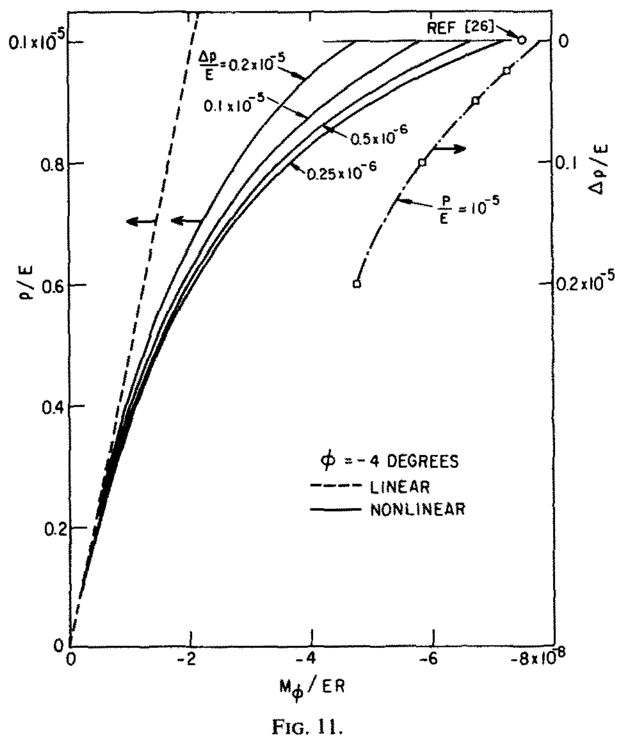

FIG. 11.

The convergence of the incremental solution is studied in Fig. 11 where the meridional moment at $\varphi = -4$ degrees is plotted for several magnitudes of the load increment. The results are rather sensitive to the size of the load increment. However, the rate of convergence is fast. The pattern of the convergence of the solution is shown on the right hand side of the Fig. 11 where for $p/E = 10^{-5}$ the meridional moments corresponding to various sizes of the load increments are plotted. It can be seen that as the size of the load increment approaches zero, the moment due to the finite element solution is just a little higher than that reported in [26].

The accuracy of the solution also depends on the number of elements. The present results were obtained by discretizing the shell into 34 elements unevenly distributed along the meridional curve. Analysis with more elements improved the results only slightly. A comparison between the results by the present curved element and the one for which u_1 is expressed by only a second order polynomial of ξ [13, 23] indicated considerable improvement in results and saving in the computer time.

CONCLUSIONS

An incremental expression of virtual work has been developed for the analysis of nonlinear problems in continuum mechanics. This expression has been reformulated in the finite element method of analysis and its linear part has been used for the large displacement analysis of elastic shells of revolution. The results of the analysis of a torus show good agreement with existing solutions.

The present method is applicable to both physically and geometrically nonlinear problems. Its application for the analysis of elastic–plastic problems has been reported in [13, 28] where more examples for both elastic and elastic–plastic shells have been presented. Another paper on the application of the above incremental analysis to elastic–plastic problems is in preparation.

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APPENDIX A

For a ring element the matrices in equation (58) are

$$\underset{4\times 8}{[B]} = \begin{bmatrix} 0 & \rho & 2\xi\rho & 3\xi^2\rho & 0 & \eta'\rho & 2\xi\eta'\rho & 3\xi^2\eta'\rho \\ \delta & \xi\delta & \xi^2\delta & \xi^3\delta & \omega & \xi\omega & \xi^2\omega & \xi^3\omega \\ 0 & -\eta'^2\Phi & -2\xi\eta'^2\Phi+\eta'\mu & -3\xi^2\eta'^2\Phi+3\mu\xi\eta' & 0 & \eta'\Phi & 2\xi\eta'\Phi-\mu & 3\xi(\xi\eta'\Phi-\mu) \\ \gamma & \eta'\Psi+\xi\gamma & 2\xi\eta'\Psi+\xi^2\gamma & 3\xi^2\eta'\Psi+\xi^3\gamma & \Gamma & -\Psi+\xi\Gamma & -2\xi\Psi+\xi^2\Gamma & -3\xi^2\Psi+\xi^3\Gamma \end{bmatrix}$$

$$\delta = \frac{\sin\psi}{r} \qquad \omega = \frac{\cos\psi}{r} \qquad \gamma = \frac{\delta\sin\varphi}{r}$$

$$\rho = \frac{1}{l(1+\eta'^2)} \qquad \Phi = \frac{\eta''}{l^2(1+\eta'^2)^{\frac{3}{2}}}$$

$$\Psi = \frac{\sin\psi + \eta'\cos\psi}{lr(1+\eta'^2)^{\frac{1}{2}}} \qquad \mu = \frac{2}{l^2(1+\eta'^2)^{\frac{3}{2}}}$$

$$\Gamma = \frac{\omega\sin\varphi}{r},$$

$$\underset{4\times 4}{[D]} = \left[\begin{array}{c|c} \underset{2\times 2}{D_{11}} & 0 \\ \hline 0 & \underset{2\times 2}{D_{22}} \end{array}\right]$$

where

$$[D_{11}] = h[c], \qquad [D_{22}] = \frac{h^3}{12}[c]$$

$$\underset{4\times 8}{[G]} = \begin{bmatrix} 0 & \eta'\rho & 2\xi\eta'\rho & 3\xi^2\eta'\rho & 0 & -\rho & -2\xi\rho & -3\xi^2\rho \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \eta'\rho & 2\xi\eta'\rho & 3\xi^2\eta'\rho & 0 & -\rho & -2\xi\rho & -3\xi^2\rho \\ 0 & \eta'\rho & 2\xi\eta'\rho & 3\xi^2\eta'\rho & 0 & -\rho & -2\xi\rho & -3\xi^2\rho \end{bmatrix}$$

$$\underset{4\times 4}{[F]} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{\Phi}{\rho} & 0 \\ 0 & 0 & 0 & \frac{\sin\varphi}{r} \end{bmatrix}$$

$$\underset{4\times 4}{[N]} = \begin{bmatrix} N_s & 0 & 0 & 0 \\ 0 & N_\theta & 0 & 0 \\ 0 & 0 & M_s & 0 \\ 0 & 0 & 0 & M_\theta \end{bmatrix}.$$

CONCLUDING NOTE

After the first version of this paper was completed, the authors' attention was drawn to related work in Refs. [29–31]. In [29], Oden has derived the finite element equations of motion for a general nonlinear continuum. However, no specific forms of these equations applicable to the present incremental solution is given. In [30], using a somewhat different approach, an incremental displacement formulation for large displacement (finite strain) problems is presented. The resulting element equilibrium equations are essentially similar to those derived here. The last reference, [31], was not available to the authors at the time of preparing this note.

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Абстракт—Начиная с общей вариационной теории приращений, для анализа геометрически и физически нелинейных задач, в механике сплошной среды, развитие теории приспособливается для упругой сплошной среды. Затем, выведенные вариацонное выражение преобразовывается в форму перемещений анализа метода конечного элемента и приспосабливается для анализа больших перемещений упругих оболочек вращения. Исследуются конвергенция и точность выведенного процесса, на очень чуствительном численном примере. Общая, представленная здесь теория приращений, имеет широкий круг применяемости и может быть использована вне специального решения этой работы.